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TRANSLATIONS CLB-3 T-638 24 February 1971

ON THE TRANSFORMATION OF A FUNCTIONAL EQUATION TO SIMPLER FORM

by

A. F. Leont'ev

translated by L. Holtschlag from

Matematicheskii Sbornik [Mathematics Symposium]
Vol. 67(109), No. 4, pp. 541-560 (1965)

SUMMARY

The function $\psi(z)$ has continuous derivatives up to and including n order on the interval $(a_1, b_1) \supset [a,b]$ of the imaginary axis, where $0 \in (a,b)$ satisfies the following equation on the interval $(a_1 - a, b_1 - b)$:

$$M(\psi) = \sum_{k=0}^{n} \int_a^b \psi^{(k)}(z+\xi)\, d\sigma_k(\xi) = 0, \tag{1}$$

where $\sigma_k(\xi)$ is a function of bounded variation on $[a,b]$. The function

$$L(\lambda) = \sum_{k=0}^{n} \lambda^k \int_a^b e^{\lambda\xi}\, d\sigma_k(\xi) \tag{2}$$

is a characteristic function of the operator M,

$$\omega(\mu, \alpha, \psi) = e^{-\alpha\mu} \sum_{k=0}^{n} \int_a^b \frac{\partial^k}{\partial\xi^k}\left[\int_0^{\xi} \psi(\alpha+\xi-\eta)\, e^{\eta\mu}\, d\eta\right] d\sigma_k(\xi), \quad \alpha \in (a_1 - a, b_1 - b). \tag{3}$$

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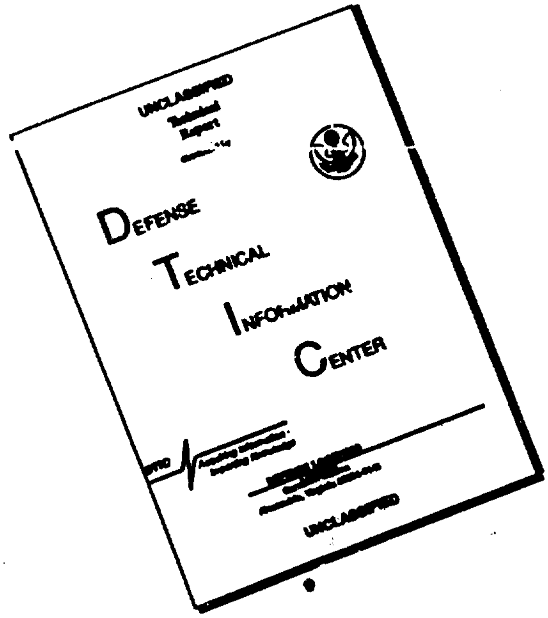
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14. KEY WORDS
Mathematics Finite Differences Functional Equations Requester: N. Rubinstein

If ρ is a zero of $L(\lambda)$, then $\tilde{L}(\lambda) = \frac{L(\lambda)}{\lambda-\rho}$, $\tilde{\sigma}_k(\xi)$ are functions whose replacement by $\sigma_k(\xi)$ in (2) maps $L(\lambda)$ into $\tilde{L}(\lambda)$, $\tilde{\omega}(\mu,\alpha,\nu)$ is a function defined by an equality of form (3) with replacement of $\sigma_k(\xi)$ by $\tilde{\sigma}_k(\nu)$, $P_\nu(z)$ is a polynomial defined by the equality

$$P_\nu(z)e^{\lambda_\nu z} = \frac{1}{2\pi i}\int_{C_\nu} \frac{\omega(\mu,\alpha,\nu)}{L(\mu)} e^{\mu z}\, d\mu,$$

where C_ν is a circle containing the single zero λ_ν of the function $L(\lambda)$; $P(z) = P_\nu(z)$ for $\lambda_\nu = \rho$. The main result of the paper is that a certain functional equation whose left side is a finite sum of Stieltjes integrals can be reduced to a simpler functional equation whose left side is a single Riemann integral. Theorems 5, 6, 7 are devoted to the summability of the series $\sum P_\nu(z)e^{\lambda_\nu(z)}$ and of its derivatives with respect to $\alpha(z)$ and its derivatives.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) The Johns Hopkins University, Applied Physics Lab. 8621 Georgia Avenue Silver Spring, Md.	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

On the Transformation of a Functional Equation to Simpler Form

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(First name, middle initial, last name)*

A. F. Leont'ev

6. REPORT DATE 24 February 1971	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
8a. CONTRACT OR GRANT NO. N00017-72-C-4401 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) CLB-3 T-638 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY NAVPLANTREPO Naval Ordnance Systems Command

13. ABSTRACT

The function $\psi(z)$ has continuous derivatives up to and including n order on the interval $(a_1, b_1) \supset [a,b]$ of the imaginary axis, where $0 \in (a,b)$ satisfies the following equation on the interval $(a_1 - a, b_1 - b)$

$$M(\psi) = \sum_{k=0}^{n} \int_a^b \psi^{(k)}(z - \xi)\, d\sigma_k(\xi) = 0, \qquad (1)$$

where $\sigma_k(\xi)$ is a function of bounded variation on $[a,b]$. The function

$$L(\lambda) = \sum_{k=0}^{n} \lambda^k \int_a^b e^{\lambda\xi}\, d\sigma_k(\xi) \qquad (2)$$

is a characteristic function of the operator M.

$$\omega(\mu, \alpha, \psi) = e^{-\alpha\mu} \sum_{k=0}^{n} \int_a^b \frac{\partial^k}{\partial\xi^k}\left[\int_0^{\xi} \psi(\alpha - \xi - \eta)\, e^{\eta\mu}\, d\eta\right] d\sigma_k(\xi), \quad \alpha \in (a_1 - a, b_1 - b). \qquad (3)$$

If ξ is a zero of $L(\lambda)$, then $\tilde{L}(\lambda) = \frac{L(\lambda)}{\lambda - \xi}$, $\tilde{\sigma}_k(\xi)$ are functions whose replacement by $\sigma_k(\xi)$ in (2) maps $L(\lambda)$ into $\tilde{L}(\lambda)$, $\tilde{\omega}(\mu, \alpha, \psi)$ is a function defined by an equality of form (3) with replacement of $\sigma_k(\xi)$ by $\tilde{\sigma}_k(\xi)$, $P_\nu(z)$ is a polynomial defined by the equality

$$P_\nu(z) e^{\lambda_\nu z} = \frac{1}{2\pi i} \int_{C_\nu} \frac{\omega(\mu, z, \psi)}{L(\mu)} e^{\mu z}\, d\mu,$$

where C_ν is a circle containing the single zero [illegible] of the function $L(\lambda)$, $F(z) = P_\nu(z)$ for [illegible]. The main result of the paper is that a certain functional equation whose left side is a finite sum of Stieltjes integrals can be reduced to a simpler functional equation whose left side is a single Riemann integral. Theorems 5, 6, 7 are devoted to the summability of the series $P_\nu(z)e^{[illegible]}$ and of its derivatives with respect to $\psi(z)$ and its derivatives.

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ON THE TRANSFORMATION OF A FUNCTIONAL EQUATION TO SIMPLER FORM*)

by

A. F. Leont'ev

Studied in the work of Delsarte, Schwartz, Kahane et al. (in this regard see [1]) were functions f(z), defined and continuous on the whole imaginary axis and satisfying on this axis the equation

$$\int_a^b f(z+\xi)\,d\sigma(\xi) = 0. \tag{1}$$

Here [a,b] is a certain segment of the imaginary axis, $\sigma(\xi)$ is a function of bounded variation on [a,b]. The functions f(z) satisfying Eq. (1) are called <u>periodic in the mean</u>. Studied in [1] were continuous solutions of Eq. (1) defined not on the whole axis, but only on a certain interval $(a_1, b_1) \supset [a,b]$.

In the present paper we consider the more general equation

$$\sum_{k=0}^{n} \int_a^b f^{(k)}(z+\xi)\,d\sigma_k(\xi) = 0, \tag{2}$$

where [a,b] is a segment of the imaginary axis, $\sigma_k(\xi)$ $(k = 0,1,\ldots,n)$ are functions of bounded variation on [a,b]. It is assumed that the function f(z) is defined and has continuous derivatives up to and including the order n on a certain interval $(a_1,b_1) \supset [a,b]$. The class of such functions is denoted by $C^n(a_1,b_1)$. A particular case of Eq. (2) is a differential-difference equation with constant coefficients. An extensive literature is devoted to differential-difference

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*) Translated from Mat. Sbornik [Math. Symposium], Vol. 67(109), No. 4, pp. 541-560 (1965).

the function

$$F(z) = f(z) - \sum_{\nu=1}^{s} P_\nu(z) e^{\lambda_\nu z}$$

in this same interval satisfies the equation

$$\int_a^b F(z+\xi)\varphi(\xi)\,d\xi = 0. \tag{3}$$

Valid also is the assertion that if the function $F(z) \in C^n(a_1, b_1)$ satisfies Eq. 3, then this function also satisfies Eq. (2).

Eq. (3) is an equation of form (1); it is even simpler than Eq. (1). Because of the result mentioned above, the solutions of Eq. (2) will have the very same properties as the solutions of Eq. (1). Several of these properties are mentioned in the paper for an example.

1. Auxiliary Assumptions

For what follows we shall need:

Lemma 1. *Let the function* $L(\lambda)$ *have the form*

$$L(\lambda) = \sum_{k=0}^{n} \lambda^k \int_a^b e^{\lambda\xi}\, d\sigma_k(\xi), \tag{1}$$

where $[a,b]$ *is the same segment of the imaginary axis, and* $\sigma_k(\xi)$ $(k = 0,1,\ldots,n)$ *are functions of bounded variation on* $[a,b]$. *Furthermore, let* β *be some zero of the function* $L(\lambda)$. *Then the function*

$$\tilde{L}(\lambda) = \frac{L(\lambda)}{\lambda - \beta}$$

has the same form as the function $L(\cdot)$, *that is*,

$$\tilde{L}(\lambda) = \sum_{k=0}^{n} \lambda^k \int_a^b e^{\lambda\xi}\, d\tilde{\sigma}_k(\xi), \tag{2}$$

where $\tilde{\sigma}_k(\xi)$ $(k = 0,1,\ldots,n)$ *are functions of bounded variation on* $[a,b]$.

Proof. Let us represent $L(\cdot)$ in the form

$$L(\lambda) = \sum_{k=0}^{n} \lambda^k \int_a^b e^{(\lambda-\beta)\xi} e^{\beta\xi}\, d\sigma_k(\xi) \tag{3}$$

and let us set

$$V_k(\xi) = \int_a^\xi e^{\beta\xi}\, d\sigma_k(\xi) \quad (k = 0, 1, \ldots, n).$$

equations. This can be found, e.g., in the book by E. Pinney [2] and the survey article by A. M. Zverkin, G.A. Kamenskii, S.B. Norkin and L.E. El'sgolts [3]. Also given in the latter are papers in which equations of type (2) are studied.

It is shown in the present paper that Eq. (2) can be reduced to Eq. (1).

Let us formulate the result in greater detail. Let the characteristic function

$$L(\lambda) = \sum_{k=0}^{n} \lambda^k \int_a^b e^{\lambda\xi}\, d\sigma_k(\xi)$$

not be identically equal to zero and have infinitely many zeros. Let $\lambda_1, \lambda_2, \ldots$ be various zeros of the function $L(\lambda)$, and let $m_1, m_2, \ldots$ be their multiplicities.

We shall assume (and this does not cause a loss in generality of the reasoning) that $[a,b]$ contains the origin strictly within itself. Let

$$\omega(\mu, f) = \sum_{k=0}^{n} \int_a^b \left[\frac{d^k}{d\xi^k} \int_0^{\xi} f(\xi - \eta)\, e^{\mu\eta} d\eta \right] d\sigma_k(\xi).$$

We associate the functions $f(z) \in C^n(a_1, b_1)$ with the series

$$f(z) \sim \sum_{\lambda_\nu} P_\nu(z)\, e^{\lambda_\nu z},$$

where

$$P_\nu(z)\, e^{\lambda_\nu z} = \frac{1}{2\pi i} \int_{C_\nu} \frac{\omega(\mu, f)}{L(\mu)} e^{\mu z}\, d\mu,$$

and C_ν is a circle with center at the point λ_ν within which there are no zeros of the function $L(\mu)$ that are different from λ_ν. We introduce the function

$$K(\mu) = \frac{L(\mu)}{(\mu - \lambda_1)^{n_1} \ldots (\mu - \lambda_s)^{n_s}},$$

where $0 \le n_\nu \le m_\nu$ $(\nu = 1, 2, \ldots, s)$. Let us select whole numbers n_ν such that the condition $n_1 + \ldots + n_s \geq n + 2$ is fulfilled. Under this condition the function $K(\mu)$ can be represented in the form

$$K(\mu) = \int_a^b \varphi(\xi)\, e^{\mu\xi}\, d\xi,$$

where $\varphi(\xi)$ is continuous on $[a,b]$. It is shown in the present article that if the function $f(z) \in C^n(a_1, b_1)$ satisfies Eq. (2) in the interval $(a_1 - a,\ b_1 - b)$,

the function

$$F(z)=f(z)-\sum_{\nu=1}^{s}P_{\nu}(z)e^{\lambda_{\nu}z}$$

in this same interval satisfies the equation

$$\int_a^b F(z+\xi)\varphi(\xi)d\xi=0. \tag{3}$$

Valid also is the assertion that if the function $F(z) \in C^n(a_1,b_1)$ satisfies Eq. 3, then this function also satisfies Eq. (2).

Eq. (3) is an equation of form (1); it is even simpler than Eq. (1). Because of the result mentioned above, the solutions of Eq. (2) will have the very same properties as the solutions of Eq. (1). Several of these properties are mentioned in the paper for an example.

1. Auxiliary Assumptions

For what follows we shall need:

Lemma 1. Let the function $L(\lambda)$ *have the form*

$$L(\lambda)=\sum_{k=0}^{n}\lambda^k\int_a^b e^{\lambda\xi}\,d\sigma_k(\xi), \tag{1}$$

where $[a,b]$ *is the same segment of the imaginary axis, and* $\sigma_k(\xi)$ $(k=0,1,\ldots,n)$ *are functions of bounded variation on* $[a,b]$. *Furthermore, let* β *be some zero of the function* $L(\lambda)$. *Then the function*

$$\widetilde{L}(\lambda)=\frac{L(\lambda)}{\lambda-\beta}$$

has the same form as the function $L(\lambda)$, *that is,*

$$\widetilde{L}(\lambda)=\sum_{k=0}^{n}\lambda^k\int_a^b e^{\lambda\xi}\,d\widetilde{\sigma}_k(\xi), \tag{2}$$

where $\widetilde{\sigma}_k(\xi)$ $(k=0,1,\ldots,n)$ *are functions of bounded variation on* $[a,b]$.

Proof. Let us represent $L(\lambda)$ in the form

$$L(\lambda)=\sum_{k=0}^{n}\lambda^k\int_a^b e^{(\lambda-\beta)\xi}e^{\beta\xi}\,d\sigma_k(\xi) \tag{3}$$

and let us set

$$V_k(\xi)=\int_a^{\xi} e^{\beta\xi}\,d\sigma_k(\xi)\quad(k=0,1,\ldots,n).$$

It is obvious that

$$V_k(a) = 0. \tag{4}$$

Applying the method of integration by parts to the integrals on the right-hand side of relation (3) and taking equality (4) into account, we get

$$L(\lambda) = \sum_{k=0}^{n} \lambda^k \left[V_k(b)\, e^{(\lambda-\beta)b} - (\lambda-\beta) \int_a^b e^{(\lambda-\beta)\xi} V_k(\xi)\, d\xi \right] = \left(\sum_{k=0}^{n} \beta^k V_k(b) \right) e^{(\lambda-\beta)b} +$$
$$+ \sum_{k=1}^{n} (\lambda^k - \beta^k) V_k(b)\, e^{(\lambda-\beta)b} - (\lambda-\beta) \sum_{k=0}^{n} \lambda^k \int_a^b e^{(\lambda-\beta)\xi} V_k(\xi)\, d\xi.$$

The first sum on the right-hand side of this equality equals zero, since it equals $L(\beta)$, and $L(\beta) = 0$. Therefore,

$$L(\lambda) = (\lambda-\beta) \left\{ \sum_{k=1}^{n} \frac{\lambda^k - \beta^k}{\lambda-\beta} V_k(b)\, e^{(\lambda-\beta)b} - \sum_{k=0}^{n} \lambda^k \int_a^b e^{(\lambda-\beta)\xi} V_k(\xi)\, d\xi \right\}.$$

Using the expansion

$$\frac{\lambda^k - \beta^k}{\lambda-\beta} = \sum_{m=0}^{k-1} \beta^{k-m-1} \lambda^m$$

and changing the order of summation, we get

$$L(\lambda) = (\lambda-\beta) \left\{ \sum_{m=0}^{n-1} \lambda^m \left[e^{(\lambda-\beta)b} \sum_{k=m+1}^{n} \beta^{k-m-1} V_k(b) - \int_a^b e^{(\lambda-\beta)\xi} V_m(\xi)\, d\xi \right] - \right.$$
$$\left. - \lambda^n \int_a^b e^{(\lambda-\beta)\xi} V_n(\xi)\, d\xi \right\}. \tag{5}$$

Let

$$\tilde{\sigma}_n(\xi) = -\int_a^{\xi} e^{-\beta\xi} V_n(\xi)\, d\xi, \quad a \leqslant \xi \leqslant b,$$

$$\tilde{\sigma}_m(\xi) = -\int_a^{\xi} e^{-\beta\xi} V_m(\xi)\, d\xi, \quad a \leqslant \xi < b,$$

$$\tilde{\sigma}_m(b) = -\int_a^{b} e^{-\beta\xi} V_m(\xi)\, d\xi + e^{-\beta b} \sum_{k=m+1}^{n} \beta^{k-m-1} V_k(b),$$

$$m = 0, 1, \ldots, n-1.$$

The functions $\tilde{\sigma}_k(\xi)$ ($k = 0,1,\ldots,n$) are functions of bounded variation on $[a,b]$. By means of these functions representation (5) is written in the form

$$L(\lambda) = (\lambda-\beta) \sum_{m=0}^{n} \lambda^m \int_a^b e^{\lambda\xi}\, d\tilde{\sigma}_m(\xi),$$

from which follows the desired representation (2). The lemma is proved.

Lemma 2. Let $L(\cdot)$ be an entire function of exponential type and let $\gamma(\xi)$ be a function Borel-associated with it, so that

$$L(\lambda) = \frac{1}{2\pi i}\int_C \gamma(\xi)\, e^{\lambda\xi}\, d\xi,$$

where C is a circle on and outside which the function $\gamma(\xi)$ is regular. Furthermore, let β be some zero of the function $L(\lambda)$. Then the function

$$\tilde{L}(\lambda) = \frac{L(\lambda)}{\lambda-\beta}$$

has the representation

$$\tilde{L}(\lambda) = \frac{1}{2\pi i}\int_C \tilde{\gamma}(\xi)\, e^{\lambda\xi}\, d\xi,$$

where

$$\tilde{\gamma}(\xi) = -e^{-\beta\xi}\int_{\xi_0}^{\xi} e^{\beta t}\gamma(t)\, dt, \quad \xi_0 \in C.$$

In order to prove this lemma we write

$$L(\lambda) = \frac{1}{2\pi i}\int_C \gamma(\xi)\, e^{\beta\xi}\, e^{(\lambda-\beta)\xi}\, d\xi.$$

After integrating by parts, setting

$$V(\xi) = \int_{\xi_0}^{\xi} e^{\beta t}\gamma(t)\, dt, \tag{6}$$

we get

$$L(\lambda) = \frac{V(\xi)}{2\pi i}\, e^{(\lambda-\beta)\xi}\Big|_C - (\lambda-\beta)\cdot\frac{1}{2\pi i}\int_C V(\xi)\, e^{(\lambda-\beta)\xi}\, d\xi. \tag{7}$$

The function $V(\xi)$ is single-valued on the contour C, since the difference between its value after going along the contour C and the value before doing this is

$$\int_C e^{\beta t}\gamma(t)\, dt = 2\pi i L(\beta) = 0.$$

Therefore, the first term on the right-hand side of equality (7) equals zero and, consequently,

$$L(\lambda) = -\frac{(\lambda-\beta)}{2\pi i}\int_C V(\xi)\, e^{(\lambda-\beta)\xi}\, d\xi = (\lambda-\beta)\cdot\frac{1}{2\pi i}\int_C \tilde{\gamma}(\xi)\, e^{\lambda\xi}\, d\xi,$$

which is what was required to be proved.

Lemma 3. Let $\psi(z)$ be an arbitrary entire function. We set

$$\omega^*(\mu,\alpha,\psi)=\frac{1}{2\pi i}e^{-\alpha\mu}\int_C\left[\int_0^{\xi}\psi(\alpha+\xi-\eta)e^{\eta\mu}\,d\eta\right]\gamma(\xi)\,d\xi,$$

$$\tilde{\omega}^*(\mu,\alpha,\psi)=\frac{1}{2\pi i}e^{-\alpha\mu}\int_C\left[\int_0^{\xi}\psi(\alpha+\xi-\eta)e^{\eta\mu}\,d\eta\right]\tilde{\gamma}(\xi)\,d\xi,$$

where the functions $\gamma(\xi)$, $\tilde{\gamma}(\xi)$ and the contour C **are** the same as in lemma 2, and α and μ are abritrary. Then there holds the equality

$$e^{\alpha\mu}\omega^*(\mu,\alpha,\psi)=(\mu-\beta)e^{\alpha\mu}\tilde{\omega}^*(\mu,\alpha,\psi)+\frac{1}{2\pi i}\int_C\psi(\alpha+\xi)\tilde{\gamma}(\xi)\,d\xi. \tag{8}$$

Let us establish this equality. We have

$$e^{\alpha\mu}\omega^*(\mu,\alpha,\psi)=\frac{1}{2\pi i}\int_C\left[\int_0^{\xi}\psi(\alpha+\eta)e^{(\xi-\eta)\mu}\,d\eta\right]\gamma(\xi)\,d\xi=$$

$$=\frac{1}{2\pi i}\int_C\left[\int_0^{\xi}\psi(\alpha+\eta)e^{-\eta\mu}e^{(\mu-\beta)\xi}\,d\eta\right]e^{\beta\xi}\gamma(\xi)\,d\xi.$$

After integrating by parts and making use of notation (6), we get

$$e^{\alpha\mu}\omega^*(\mu,\alpha,\psi)=\left[\frac{1}{2\pi i}V(\xi)\int_0^{\xi}\psi(\alpha+\eta)e^{-\eta\mu}e^{(\mu-\beta)\xi}\,d\eta\right]\Big|_C-$$

$$-\frac{1}{2\pi i}\int_C\left[\psi(\alpha+\xi)+(\mu-\beta)\int_0^{\xi}\psi(\alpha+\eta)e^{(\xi-\eta)\mu}\,d\eta\right]e^{-\beta\xi}V(\xi)\,d\xi.$$

Included in the first bracketed expression is a function single-valued on the contour C, since $V(\xi)$ is a single-valued function. Therefore, the first term on the right-hand side of the last equality equals zero and, consequently,

$$e^{\alpha\mu}\omega^*(\mu,\alpha,\psi)=(\mu-\beta)\cdot\frac{1}{2\pi i}\int_C\left[\int_0^{\xi}\psi(\alpha+\eta)e^{(\xi-\eta)\mu}\,d\eta\right]\tilde{\gamma}(\xi)\,d\xi+$$

$$+\frac{1}{2\pi i}\int_C\psi(\alpha+\xi)\tilde{\gamma}(\xi)\,d\xi.$$

Because

$$\int_0^{\xi}\psi(\alpha+\eta)e^{(\xi-\eta)\mu}\,d\eta=\int_0^{\xi}\psi(\alpha+\xi-\eta)e^{\eta\mu}\,d\eta,$$

the first integral on the right-hand side equals $e^{\alpha\mu}\tilde{\omega}^*(\mu,\alpha,\psi)$. Hence, equality (8) is indeed valid.

The functions $\omega^*(\mu,\alpha,\psi)$ and $\tilde{\omega}^*(\mu,\alpha,\psi)$ are generated by the functions $L(\xi)$ and $\tilde{L}(\xi)$ from lemma 2, respectively. Let us introduce the functions $\omega(\mu,\alpha,\psi)$

$\widetilde{\omega}(\mu,\alpha,\psi)$, which are generated by the functions $L(\lambda)$ and $\widetilde{L}(\lambda)$ of lemma 1, and let us set up a relation for them that is analogous to relation (8).

Lemma 4. We set

$$\omega(\mu,\alpha,\psi)=e^{-\alpha\mu}\sum_{k=0}^{n}\int_{a}^{b}\frac{\partial^k}{\partial\xi^k}\left[\int_{0}^{\xi}\psi(\alpha+\xi-\eta)e^{\eta\mu}\,d\eta\right]d\sigma_k(\xi),$$

$$\widetilde{\omega}(\mu,\alpha,\psi)=e^{-\alpha\mu}\sum_{k=0}^{n}\int_{a}^{b}\frac{\partial^k}{\partial\xi^k}\left[\int_{0}^{\xi}\psi(\alpha+\xi-\eta)e^{\eta\mu}\,d\eta\right]d\widetilde{\sigma}_k(\xi),$$

where the functions $\sigma_k(\xi)$, $\widetilde{\sigma}_k(\xi)$ are the same as in lemma 1. Let the segment $[a,b]$ contain the origin rigorously within itself and let the function $\psi(z)$ be defined and have continuous derivatives up to and including n order on a certain interval $(a_1,b_1)\supset[a,b]$. Then, for any μ and $\alpha\in(a_1-a,\ b_1-b)$ there holds the equality

$$e^{\alpha\mu}\omega(\mu,\alpha,\psi)=(\mu-\beta)e^{\alpha\mu}\widetilde{\omega}(\mu,\alpha,\psi)+\sum_{k=0}^{n}\int_{a}^{b}\psi^{(k)}(\alpha+\xi)\,d\widetilde{\sigma}_k(\xi). \tag{9}$$

We first verify equality (9) for the function $\psi(z)=e^{\lambda z}$, where λ is an arbitrary number. For such a function

$$\int_{0}^{\xi}\psi(\alpha+\xi-\eta)e^{\eta\mu}\,d\eta=e^{\alpha\lambda}\frac{e^{\mu\xi}-e^{\lambda\xi}}{\mu-\lambda},$$

therefore,

$$e^{\alpha\mu}\omega(\mu,\alpha,\psi)=e^{\alpha\lambda}\frac{L(\mu)-L(\lambda)}{\mu-\lambda},\qquad e^{\alpha\mu}\widetilde{\omega}(\mu,\alpha,\psi)=e^{\alpha\lambda}\frac{\widetilde{L}(\mu)-\widetilde{L}(\lambda)}{\mu-\lambda}.$$

Moreover, if $\psi(z)=e^{\lambda z}$, then

$$\sum_{k=0}^{n}\int_{a}^{b}\psi^{(k)}(\alpha+\xi)\,d\widetilde{\sigma}_k(\xi)=e^{\alpha\lambda}\widetilde{L}(\lambda).$$

From these equalities it follows that relation (9) for the function $\psi(z)=e^{\lambda z}$ is indeed valid.

Let us examine the system $\{e^{imz}\}$. It is complete in any vertical strip of width less than 2π. Any function analytic in this strip can be approximated arbitrarily well by means of finite linear combinations of functions from the indicated system. Consequently, relation (9) is valid for any analytic function from the strip; in particular, it is valid for polynomials. Let $\psi(z)$ be an

arbitrary function defined on the interval $(a_1, b_1) \supset [a,b]$ and having continuous derivatives of up to n order inclusive on this interval. We choose the segment $[a_2, b_2]$ such that $[a,b] \subset [a_2,b_2] \subset (a_1,b_1)$. Let P(z) be a polynomial having the property that

$$|\psi^{(k)}(z) - P^{(k)}(z)| < \varepsilon, \quad z \in [a_2, b_2] \quad (k = 0, 1, 2, \ldots, n). \tag{10}$$

where $\varepsilon > 0$ is an arbitrary number. Such a polynomial exists. Since relation (9) is valid for the polynomial P(z), we conclude on the basis of (10) that it will also be valid for the function $\psi(z)$ when $\alpha \in (a_2 - a,\ b_2 - b)$. The lemma is proved.

<u>Remark</u>. Lemmas 2 and 3 make it possible to get equality (8) in a natural and very simple manner; they are not used in the following. Equality (8), however, suggested the idea of the existence of equality (9).

<u>Lemma 5. Let $\omega(\mu,\alpha,\psi)$ be the function defined in lemma 4 and let</u> $L(\lambda)$ <u>be the function defined in lemma 1. If the function $\psi(z)$ is defined, has continuous derivatives of up to n order inclusive in the interval</u> $(a_1,b_1) \supset [a,b]$, <u>and in the interval</u> $(a_1 - a,\ b_1 - b)$ <u>satisfies the equation</u>

$$\sum_{k=0}^{n} \int_a^b \psi^{(k)}(z+\xi)\, d\sigma_k(\xi) \doteq 0, \tag{11}$$

<u>then the value of the integral</u>

$$\int_\Gamma \frac{\omega(\mu,\alpha,\psi)}{L(\mu)} e^{\mu z}\, d\mu, \tag{12}$$

<u>where Γ is any closed contour on which</u> $L(\mu) \neq 0$, <u>does not depend on</u> α, $\alpha \in (a_1 - a,\ b_1 - b)$.

We have

$$\omega = \omega(\mu,\alpha,\psi) = \sum_{k=0}^{n} \int_a^b \frac{\partial^k}{\partial \xi^k}\left[\int_0^\xi \psi(\alpha+\xi-\eta)\, e^{\mu(\eta-\alpha)}\, d\eta\right] d\sigma_k(\xi) =$$

$$= \sum_{k=0}^{n} \int_a^b \frac{\partial^k}{\partial \xi^k}\left[\int_{-\alpha}^{\xi-\alpha} \psi(\xi - t)\, e^{\mu t}\, dt\right] d\sigma_k(\xi),$$

on the basis of which

$$\frac{\partial \omega}{\partial \alpha} = \sum_{k=0}^{n} \int_a^b \frac{\partial^k}{\partial \xi^k}\left[\frac{\partial}{\partial \alpha}\int_{-\alpha}^{\xi-\alpha} \psi(\xi - t)\, e^{\mu t}\, dt\right] d\sigma_k(\xi).$$

Since

$$\frac{\partial}{\partial\alpha}\int_{-\alpha}^{\xi-\alpha}\psi(\xi-t)\,e^{\mu t}\,dt=-\psi(\alpha)\,e^{\mu(\xi-\alpha)}+\psi(\xi+\alpha)\,e^{-\alpha\mu},$$

we find that

$$\frac{\partial\omega}{\partial\alpha}=-\psi(\alpha)\,e^{-\alpha\mu}L(\mu)+e^{-\alpha\mu}\sum_{k=0}^{n}\int_a^b\psi^{(k)}(\xi+\alpha)\,d\sigma_k(\xi). \tag{13}$$

By virtue of condition (11), the second term on the right-hand side of this equality equals zero; therefore,

$$\frac{\partial\omega}{\partial\alpha}=-\psi(\alpha)\,e^{-\alpha\mu}L(\mu).$$

Hence it follows that

$$\frac{\partial}{\partial\alpha}\int_\Gamma\frac{\omega(\mu,\alpha,\psi)}{L(\mu)}e^{\mu z}\,d\mu=-\psi(\alpha)\int_\Gamma e^{\mu(z-\alpha)}\,d\mu=0,$$

which is what was required to be proved.

Lemma 6. If the value of integral (12), where Γ is any closed contour on which $L(\mu)\neq 0$, does not depend on α and the function $L(\mu)$ has at least one zero, the function $\psi(z)$ satisfies Eq. (11).

According to the condition and equality (13), we have

$$\frac{\partial}{\partial\alpha}\int_\Gamma\frac{\omega(\mu,\alpha,\psi)}{L(\mu)}e^{\mu z}\,d\mu=-\psi(\alpha)\int_\Gamma e^{\mu(z-\alpha)}\,d\mu+$$

$$+\left\{\sum_{k=0}^{n}\int_a^b\psi^{(k)}(\xi-\alpha)\,d\sigma_k(\xi)\right\}\int_\Gamma\frac{e^{\mu(z-\alpha)}}{L(\mu)}\,d\mu=0.$$

The first term equals zero and the integral

$$\int_\Gamma\frac{e^{\mu(z-\alpha)}}{L(\mu)}\,d\mu,$$

is not identically equal to zero for any z if as Γ we take a circle of small radius with center at the zero μ_0 of the function $L(\mu)$. Therefore,

$$\sum_{k=0}^{n}\int_a^b\psi^{(k)}(\xi-\alpha)\,d\sigma_k(\xi)=0.$$

Lemma 7. Let μ_0 be a zero (of multiplicity m) of the function $L(\mu)$ and let $\psi(z)=z^pe^{\mu_0 z}$, where $0\leq p<m$. Then the residue of the function

$$\frac{\omega(\mu,\alpha,\psi)}{L(\mu)}e^{\mu z}$$

(as a function of the variable μ) equals $\psi(z)$ at the point $\mu = \beta$ and equals zero at all the zeros of the function $L(\mu)$ that are different from β.

Proof. In the proof of lemma 4 it was shown that

$$\omega(\mu, \alpha, e^{\lambda z}) = e^{\alpha(\lambda-\mu)} \frac{L(\mu) - L(\lambda)}{\mu - \lambda}.$$

Hence,

$$\omega(\mu, \alpha, \psi) = \frac{\partial^p}{\partial\lambda^p} \left[e^{\alpha(\lambda-\mu)} \frac{L(\mu) - L(\lambda)}{\mu - \lambda} \right] \quad (\lambda = \beta, \ \psi(z) = z^p e^{\beta z})$$

or

$$\omega(\mu, \alpha, \psi) = \sum_{\nu=0}^{p} C_p^\nu \alpha^\nu e^{\alpha(\lambda-\mu)} \frac{\partial^{p-\nu}}{\partial\lambda^{p-\nu}} \left[\frac{L(\mu) - L(\lambda)}{\mu - \lambda} \right] \quad (\lambda = \beta).$$

But

$$\frac{\partial^k}{\partial\lambda^k} \left[\frac{L(\mu)}{\mu - \lambda} - \frac{L(\lambda)}{\mu - \lambda} \right]_{\lambda=\beta} = \left\{ \frac{k!\, L(\mu)}{(\mu - \lambda)^{k+1}} - \sum_{j=1}^{k} C_k^j \frac{(k-j)!\, L^{(j)}(\lambda)}{(\mu - \lambda)^{k-j+1}} \right\}_{\lambda=\beta} =$$

$$= \frac{k!\, L(\mu)}{(\mu - \lambda)^{k+1}}.$$

Therefore,

$$\omega(\mu, \alpha, \psi) = \left(\sum_{\nu=0}^{p} C_p^\nu \alpha^\nu \frac{(p-\nu)!}{(\mu - \beta)^{p-\nu+1}} \right) L(\mu) e^{\alpha(\beta-\mu)}$$

and consequently

$$\frac{\omega(\mu, \alpha, \psi)}{L(\mu)} e^{\mu z} = e^{\beta z} e^{(\mu-\beta)(z-\alpha)} \sum_{\nu=0}^{p} C_p^\nu \alpha^\nu \frac{(p-\nu)!}{(\mu - \beta)^{p-\nu+1}}. \tag{14}$$

Hence, we conclude: at points $\mu \neq \beta$ function (14) is regular and therefore its residue at these points equals zero; at the point $\mu = \beta$ the residue equals the coefficient c_{-1} for $(\mu - \beta)^{-1}$ in the expansion of function (14) into a Laurent series; in which case

$$c_{-1} = e^{\beta z} \sum_{\nu=0}^{p} C_p^\nu \alpha^\nu \frac{(p-\nu)!\,(z-\alpha)^{p-\nu}}{(p-\nu)!} = e^{\beta z} [\alpha + (z - \alpha)]^p = z^p e^{\beta z}.$$

The lemma is proved.

2. Main Results

Considered in this section are the functions $\psi(z)$, defined and having continuous derivatives of up to n order inclusive on the interval $(a_1,b_1) \supset [a,b]$ of the imaginary axis and solving in the interval $(a_1 - a,\ b_1 - b)$ the equation

$$M(\psi) = \sum_{k=0}^{n} \int_a^b \psi^{(k)}(z+\xi)\,d\sigma_k(\xi) = 0, \tag{1}$$

where $\sigma_k(\xi)$ $(k = 0,1,\ldots,n)$ are functions of bounded variation on $[a,b]$. We assume that the origin lies rigorously within the segment $[a,b]$.

The function

$$L(\lambda) = \sum_{k=0}^{n} \lambda^k \int_a^b e^{\lambda\xi}\,d\sigma_k(\xi) \tag{2}$$

is called a <u>characteristic function corresponding to Eq. (1)</u>. We note that for any λ

$$M(e^{\lambda z}) = L(\lambda)e^{\lambda z}. \tag{3}$$

When $L(\lambda) \equiv 0$, Eq. (1), according to equality (3), has a solution $\psi(z) = e^{\lambda z}$, where λ is any number. Hence it follows that Eq. (1) is solved by any function having continuous derivatives of up to and including n order on (a_1,b_1). In this case Eq. (1) reduces essentially to an identity, and therefore this case is not interesting.

Let us examine the case when the function $L(\lambda)$ has no zeros at all. Then, since $L(\lambda)$ is an exponential function, $L(\lambda) = Ae^{\nu\lambda}$, where A and ν are constants, $A \neq 0$. Since, as is evident from representation (2), the function $L(\lambda)$ behaves on the real axis as $O(|\lambda|^n)$, the number ν is purely imaginary. Furthermore, it is also evident from representation (2) that

$$|L(\lambda)| < O(|\lambda|^n)e^{|a|\cdot|\lambda|},\ \operatorname{Im}\lambda > 0;\quad |L(\lambda)| < O(|\lambda|^n)e^{|b|\cdot|\lambda|},\ \operatorname{Im}\lambda < 0$$

on the imaginary axis for large $|\lambda|$. Therefore, the number ν belongs to the segment $[a,b]$. According to equality (3) we have

$$M(e^{\lambda z}) = Ae^{\gamma\lambda}e^{\lambda z}. \tag{4}$$

We set $M_1(\psi) = A\psi(z+\gamma)$. On the basis of (4) we conclude that $M(e^{\lambda z}) = M_1(e^{\lambda z})$ for any λ. Hence it follows that for any functions $\psi(z)$ having continuous derivatives of up to n order on (a_1, b_1), the following relation is valid:

$$M(\psi) = M_1(\psi) = A\psi(z+\gamma).$$

Consequently, in this case Eq. (1) has only the trivial solution $\psi(z) \equiv 0$.

Now let the function $L(\lambda)$ have a finite number of zeros $\lambda_1, \lambda_2, \ldots, \lambda_s$ whose multiplicities are equal respectively to $m_1, \ldots, m_s$. We have

$$L(\lambda) = P(\lambda)e^{\gamma\lambda}, \quad P(\lambda) = A(\lambda-\lambda_1)^{m_1} \ldots (\lambda-\lambda_s)^{m_s}, \quad \gamma \in [a, b].$$

It is obvious that $N = m_1 + \ldots + m_s \leq n$. Let $P(\lambda) = \sum_{\nu=0}^{N} a_\nu \lambda^\nu$. We set $M_1(\psi) = \sum_{\nu=0}^{N} a_\nu \psi^{(\nu)}(z+\gamma)$. As in the preceding case, it can be seen that the equality $M(\psi) = M_1(\psi)$ for the functions $\psi(z)$ being considered. The solution of Eq.(1) reduces to the solution of the equation $M_1(\psi) = 0$. The general solution of the latter has the form

$$\psi(z) = \sum_{\nu=1}^{s} P_\nu(z)e^{\lambda_\nu z},$$

where $P_\nu(z)$ is an arbitrary polynomial of degree less than m_ν.

These cases are not of great interest. Let us now go on to the case when $L(\lambda) \not\equiv 0$ and $L(\lambda)$ has infinitely many zeros. Let $\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots$ be various zeros of the function $L(\lambda)$ and let $m_1, m_2, \ldots, m_k, \ldots$, resp., be their multiplicities.

We associate with the solution $\psi(z)$ of Eq.(1) the series

$$\psi(z) \sim \sum_{\lambda_\nu} P_\nu(z)e^{\lambda_\nu z}, \tag{5}$$

where

$$P_\nu(z)e^{\lambda_\nu z} = \frac{1}{2\pi i}\int_{C_\nu} \frac{\omega(\mu, \alpha, \psi)}{L(\mu)} e^{\mu z}\, d\mu.$$

Here C_ν is a circle with center at the point λ_ν, within which there are no zeros of the function $L(\mu)$ that are different from λ_ν and

$$\omega(\mu, \alpha, \psi)=e^{-\alpha\mu} \sum_{k=0}^{n} \int_{a}^{b} \frac{\partial^k}{\partial \xi^k}\left[\int_{0}^{\xi}\psi(\alpha+\xi-\eta)\, e^{\eta\mu} d\eta\right] d\sigma_k(\xi), \quad \alpha \in (a_1-a,\, b_1-b).$$

It is obvious that $P_\nu(z)$ is a polynomial of degree less than m_ν. According to lemma 5 the polynomial $P_\nu(z)$ does not depend on the parameter α.

We note that for a function $\psi(z)$ of the form

$$\psi(z) = \sum_{\nu=1}^{N} P_\nu(z)\, e^{\lambda_\nu z}, \tag{6}$$

where $P_\nu(z)$ is an arbitrary polynomial of degree less than m_ν, series (5) coincides with the finite sum (6) by virtue of lemma 7. This circumstance explains why series (5) is chosen as the series corresponding to the solution of Eq.(1).

Let β be some zero of the function $L(\mu)$ ($\beta = \lambda_\nu$ for a certain ν). Along with Eq.(1) let us consider the equation

$$M_1(\psi) \equiv \sum_{k=0}^{n} \int_{a}^{b} \psi^{(k)}(z+\xi)\, d\tilde{\sigma}_k(\xi) = 0, \tag{7}$$

whose characteristic function

$$\tilde{L}(\mu) = \frac{L(\mu)}{\mu-\beta}$$

is defined in lemma 1. If $\beta = \lambda_\nu$, the term $P_\nu(z)e^{\lambda_\nu z}$ of series (5) corresponding to this zero β will be denoted by $P(z)e^{\beta z}$.

Theorem 1. Let $\psi(z)$ be a solution of Eq.(1). Then the function

$$f(z) = \psi(z) - P(z)\, e^{\beta z} \tag{8}$$

is the solution of Eq.(7). Moreover, there holds the equality

$$\frac{\omega(\mu, \alpha, f)}{L(\mu)} = \frac{\tilde{\omega}(\mu, \alpha, f)}{\tilde{L}(\mu)}, \tag{9}$$

where the function $\tilde{\omega}(\mu,\alpha,f)$ was introduced in lemma 4 and is constructed by means of the function $\tilde{L}(\mu)$ in the same way as the function $\omega(\mu,\alpha,f)$ is constructed by means of $L(\mu)$.

In order to prove the theorem, we note first that by lemma 7

$$\frac{1}{2\pi i}\int_C \frac{\omega(\mu,\alpha,\varphi)}{L(\mu)} e^{\mu z} d\mu = \varphi(z), \quad \varphi(z) = P(z) e^{\beta z},$$

where C is a circle with center at the point β, within which there are no other zeros of the function $L(\mu)$ except the point β. On this basis, since $f = \psi - \varphi$, we have

$$\frac{1}{2\pi i}\int_C \frac{\omega(\mu,\alpha,f)}{L(\mu)} e^{\mu z} d\mu = \frac{1}{2\pi i}\int_C \frac{\omega(\mu,\alpha,\psi)}{L(\mu)} e^{\mu z} d\mu - \frac{1}{2\pi i}\int_C \frac{\omega(\mu,\alpha,\varphi)}{L(\mu)} e^{\mu z} d\mu = 0.$$

Hence we conclude that the point β is, for the entire function $\omega(\mu,\alpha,f)$, a zero of multiplicity equal at least to m, where m is the multiplicity of the zero $\mu = \beta$ of the function $L(\mu)$. By virtue of lemma 4 there holds the relation

$$e^{\alpha\mu}\omega(\mu,\alpha,f) = (\mu-\beta)e^{\alpha\mu}\tilde{\omega}(\mu,\alpha,f) + \sum_{k=0}^{n}\int_a^b f^{(k)}(\alpha+\xi)\,d\tilde{\sigma}_k(\xi),$$
$$\alpha \in (a_1 - a,\ b_1 - b). \tag{10}$$

If here we set $\mu = \beta$, we get

$$\sum_{k=0}^{n}\int_a^b f^{(k)}(\alpha+\xi)\,d\tilde{\sigma}_k(\xi) = 0, \quad \alpha \in (a_1 - a,\ b_1 - b). \tag{11}$$

Consequently, the function f(z) actually satisfies Eq.(7). On the basis of (11) the equality (10) takes the form

$$e^{\alpha\mu}\omega(\mu,\alpha,f) = (\mu-\beta)e^{\alpha\mu}\tilde{\omega}(\mu,\alpha,f),$$

from which follows relation (9). The theorem is proved.

Theorem 2. If the function $\psi(z)$ satisfies Eq.(7), then it also satisfies Eq.(1).

Proof. We make use of lemma 4, according to which

$$e^{\alpha\mu}\omega(\mu,\alpha,\psi) = (\mu-\beta)e^{\alpha\mu}\tilde{\omega}(\mu,\alpha,\psi) + \sum_{k=0}^{n}\int_a^b \psi^{(k)}(\alpha+\xi)\,d\tilde{\sigma}_k(\xi),$$
$$\alpha \in (a_1 - a,\ b_1 - b).$$

By the condition of the theorem the second term on the right-hand side of this equality equals zero; therefore, $\dfrac{\omega(\mu,\alpha,\psi)}{L(\mu)} = \dfrac{\tilde{\omega}(\mu,\alpha,\psi)}{\tilde{L}(\mu)}$

and

$$\int_{\Gamma} \frac{\omega(\mu, \alpha, \psi)}{L(\mu)} e^{\mu z} d\mu = \int_{\Gamma} \frac{\widetilde{\omega}(\mu, \alpha, \psi)}{\widetilde{L}(\mu)} e^{\mu z} d\mu, \qquad (12)$$

where Γ is any closed contour on which $L(\mu) \neq 0$. According to lemma 5, since the function $\psi(z)$ satisfies Eq.(7), the right-hand side of relation (12) does not depend on α. Consequently, the left-hand side of this relation also does not depend on α. Then, by lemma 6 the function $\psi(z)$ satisfies Eq.(1), which is what had to be proved.

The function $\omega(\mu,\alpha,\psi)$, constructed by means of the function $L(\mu)$, will be denoted by $\omega_L(\mu,\alpha,.)$. In the new notation the function $\widetilde{\omega}(\mu,\alpha,\psi)$ is $\omega_{\widetilde{L}}(\mu,\alpha,\psi)$. The left side of the equation

$$\sum_{k=0}^{n} \int_{a}^{b} \psi^{(k)}(z+\xi)\, d\sigma_k(\xi) = 0,$$

for which the characteristic function is

$$L(\lambda) = \sum_{k=0}^{n} \lambda^k \int_{a}^{b} e^{\lambda\xi} d\sigma_k(\xi),$$

will be denoted by $M_L(.) = M_L[\psi(z)]$.

In the proof of theorem 1 it was shown that the point β for the function $\omega(\mu,\alpha,f)$ is a zero of multiplicity not less than m. From relation (9) we conclude that this point is a zero of the function $\widetilde{\omega}(\mu,\alpha,f)$, whose multiplicity is not less than m - 1. Let m > 1. We set

$$L_2(\mu) = \frac{L(\mu)}{(\mu-\beta)^2} = \frac{L_1(\mu)}{\mu-\beta}, \quad L_1(\mu) = \widetilde{L}(\mu)$$

according to lemma 4

$$e^{\alpha\mu}\omega_{L_1}(\mu, \alpha, f) = (\mu-\beta)e^{\alpha\mu}\omega_{L_2}(\mu, \alpha, f) + M_{L_2}[f(\alpha)], \qquad \alpha \in (a_1 - a, \; b_1 - b).$$

Setting here $\mu = \beta$, we find that $M_{L_2}[f(\alpha)] = 0$ and

$$\frac{\omega_{L_1}(\mu, \alpha, f)}{L_1(\mu)} = \frac{\omega_{L_2}(\mu, \alpha, f)}{L_2(\mu)}.$$

The point β for $\omega_{L_2}(\mu,\alpha,f)$ will be a zero whose multiplicity is not less than $m-2$. If $m > 2$, the indicated process can be continued. As a result, we reach the conclusion that the function $f(z) = \psi(z) - P(z)e^{\beta z}$ satisfies the equation $M_{L_k}(f) = 0$ and that the following relation is valid:

$$\frac{\omega_L(\mu,\alpha,f)}{L(\mu)} = \frac{\omega_{L_k}(\mu,\alpha,f)}{L_k(\mu)}; \quad L_k(\mu) = \frac{L(\mu)}{(\mu-\beta)^k}, \quad 0 < k \leqslant m. \tag{13}$$

We note that on the basis of relation (13) the series $f(z) \sim \sum' P_\nu(z)e^{\lambda_\nu z}$ corresponds to the function $f(z)$ as the solution of the equation $M_{L_k}(f) = 0$; this series differs from series (5) only in that the term $P(z)e^{\beta z}$, corresponding in series (5) to the zero β of the function $L(\mu)$, is absent in it.

The above reasoning has made it possible to go from an equation with a characteristic function $L(\mu)$ to an equation with a characteristic function $L_k(\mu)$. If γ ($\gamma \neq \beta$) is a zero (of multiplicity p) of the function $L_k(\mu)$, it is possible to go over in a similar way from an equation with a characteristic function $L_k(\mu)$ to an equation with a characteristic function

$$\frac{L_k(\mu)}{(\mu-\gamma)^q}, \quad 0 < q \leqslant p.$$

To sum up, we can formulate the following theorem.

Theorem 3. Let $\psi(z)$ be the solution of Eq. (1) to which the series (5) corresponds. Then the function

$$F(z) = \psi(z) - \sum_{\nu=1}^{s} P_\nu(z)e^{\lambda_\nu z}$$

is the solution of the equation

$$M_K[F(z)] = 0, \quad z \in (a_1 - a,\ b_1 - b) \tag{14}$$

with the characteristic function

$$K(\mu) = \frac{L(\mu)}{(\mu-\lambda_1)^{n_1}\ldots(\mu-\lambda_s)^{n_s}}, \quad 0 < n_1 \leqslant m_1, \ \ldots, \ 0 < n_s \leqslant m_s, \tag{15}$$

and

$$\frac{\omega_L(\mu,\alpha,F)}{L(\mu)} = \frac{\omega_K(\mu,\alpha,F)}{K(\mu)}, \quad \alpha \in (a_1 - a,\ b_1 - b). \tag{16}$$

By successive application of theorem 2 it is possible to conclude that if $F(z)$ is a solution of Eq. (14), then $F(z)$ will also be a solution of the original equation (1).

We note that by virtue of lemma (1) the function $K(\mu)$ has the form

$$K(\mu) = \sum_{k=0}^{n} \mu^k \int_a^b e^{\mu\xi} \, d\sigma_k'(\xi),$$

where $\sigma_k'(\xi)$ are functions of bounded variation on the segment $[a,b]$, and the number n is here the same as in formula (1). In conformity with this,

$$M_k(F) = \sum_{k=0}^{n} \int_a^b F^{(k)}(z+\xi) \, d\sigma_k'(\xi). \tag{17}$$

Let us assume that the function $F(z)$ has continuous derivatives of up to and including n order on the interval $(a_1,b_1) \supset [a,b]$. The class of such functions will be denoted by $C^n(a_1,b_1)$.

Let us find a simpler representation for the operator (17).

For large $|\mu|$ the function $L(\mu)$ varies on the real axis as $O(|\mu|^n)$. In representation (15) we choose s and $n_1,\ldots,n_s$ under the condition: $n_1 + \ldots + n_s = n + 2$. Then the function $K(\mu)$ will vary on the real axis as $O(|\mu|^{-2})$. On the imaginary axis, for large $|\mu|$ we have

$$|K(\mu)| < O(|\mu|^{-2}) e^{|a|\cdot|\mu|}, \quad \operatorname{Im}\mu > 0;$$
$$|K(\mu)| < O(|\mu|^{-2}) e^{|b|\cdot|\mu|}, \quad \operatorname{Im}\mu < 0.$$

Let the segment $[a,b]$ contain the origin so that Im a 0 Im b. Let $\gamma(t)$ be a function that is Borel-associated with the function $K(\mu)$. We have

$$\gamma(t) = \int_0^\infty K(\mu) e^{-\mu t} \, d\mu, \tag{18}$$

$$K(\mu) = \frac{1}{2\pi i} \int_C \gamma(t) e^{\mu t} \, dt, \tag{19}$$

where C is a closed contour encompassing all the singularities of the function $\gamma(t)$. If in integral (18) we choose the positive real semi-axis as the path of

integration, then, taking into account the above-mentioned behavior of $K(\mu)$ on this semi-axis we find that the function $\gamma(t)$ is regular in the half-plane $\mathrm{Re}(t) < 0$ and is continuous in the closed half-plane $\mathrm{Re}\,(t) \leq 0$. Analogously, we see that $\gamma(t)$ is regular in the half-plane $\mathrm{Re}\,(t) > 0$ and is continuous in the closed half-plane $\mathrm{Re}\,(t) \geq 0$. If as the path of integration in integral (18) we first choose the upper part of the imaginary axis, and then the lower part, we find that the function $\gamma(t)$ is regular in the half-plane $\mathrm{Im}\,(t) > |b|$ and $\mathrm{Im}\,(t) < -|a|$ and is continuous in the closed half-planes. Keeping this in mind, from (19) we get

$$K(\mu) = \int_a^b \varphi(\xi) e^{\mu\xi} d\xi, \tag{20}$$

where $\varphi(\xi) = \frac{1}{2\pi i}\left[\gamma(\xi + 0) - \gamma(\xi - 0)\right]$, and $\gamma(\xi + 0)$ is the limit of the function $\gamma(t)$ as the point t tends to the right to the point ξ of the segment $[a,b]$, and $\gamma(\xi - 0)$ is the limit of the function $\gamma(t)$ as the point t tends to the left to the point ξ. The function $\varphi(\xi)$ is continuous on the segment $[a,b]$ and equals zero on the imaginary axis outside this segment. We note that the function $\varphi(\xi)$ is not everywhere equal to zero on $[a,b]$, because if it were, the function $K(\mu)$, and consequently also the function $L(\mu)$, would be identically equal to zero, which contradicts our condition. We set

$$A(F) = \int_a^b \varphi(\xi) F(z + \xi) d\xi, \quad F(z) \in C^n(a_1, b_1).$$

The equality $A(F) = M_K(F)$ is fulfilled for the function $F(z) = e^{\lambda z}$ with arbitrary λ. Hence it follows that the latter equality also holds for arbitrary functions $F(z)$ from the class $C^n(a_1,b_1)$. Thus,

$$M_K(F) \equiv \int_a^b \varphi(\xi) F(z + \xi) d\xi, \quad F(z) \in C^n(a_1, b_1). \tag{21}$$

This makes it possible to formulate the following theorem on the basis of theorem 3.

<u>Theorem 4</u>. <u>Let</u> [illegible](z) <u>be the solution of Eq.(1) to which series</u> (5) <u>corres-</u>

ponds. Then the function

$$F(z) \quad \psi(z) - \sum_{\nu=1}^{s} P_\nu(z) e^{\lambda_\nu z},$$

where s is chosen such that the condition $m_1 + \ldots + m_s \geq n + 2$ is fulfilled, satisfies Eq.(21). Conversely, if the function $F(z) \in C^n(a_1,b_1)$ satisfies Eq.(21), it also satisfies Eq.(1).

Theorem 4 permits reduction of the question as to the solution of Eq.(1) to the question of the solution of the simpler equation (21).

Eq.(21) was studied in detail in [1]. Let us give some of the results from this paper. Let $[a_2,b_2] \subset [a,b]$ be the least segment outside which the function $\varphi(\xi)$ in representation (21) equals zero. Without loss of generality of the reasoning, it can be assumed that it is symmetrical relative to the origin, so that $a_2 = -qi$, $b_2 = qi$, $q > 0$. If we set $\sigma(\xi) = \int_{-qi}^{\xi} \varphi(\xi)\,d\xi$, then Eq.(21) can be represented in the form

$$M_K(F) = \int_{-qi}^{qi} F(z+\xi)\,d\sigma(\xi) = 0, \tag{21'}$$

and the characteristic function (20) in the form

$$K(\mu) = \int_{-qi}^{qi} e^{\mu\xi}\,d\sigma(\xi).$$

The function $K(\mu)$ has the following properties:

1) for almost all $\varphi \in [0,2\pi]$ there exists

$$\lim_{r\to\infty} \frac{\ln|K(re^{i\varphi})|}{r} = q|\sin\varphi|; \tag{22}$$

2) there exists a sequence of numbers $\rho_k > 0$ ($\rho_k \uparrow \infty$) and a number $p > 0$ such that

$$\ln|K(re^{i\varphi})| > (q|\sin\varphi| - \varepsilon)r, \quad \rho_k - p \leqslant r \leqslant \rho_k + p, \quad k > N(\varepsilon), \tag{23}$$

where $\varepsilon > 0$ is any number.

Let $\beta > 0$ be a sufficiently small number such that on the rays

$\arg \mu = \frac{\pi}{2} + \beta$ and $\arg \mu = -(\frac{\pi}{2} + \beta)$ relation (22) is fulfilled. We assume that the function $F(z)$ is continuous on the interval $(a_1, b_1) \supset [-qi, qi]$ and satisfies Eq.(21) for $z \in (a_1 + qi,\ b_1 - qi)$. Let μ_ν ($\nu = 1,2,\ldots$) be various zeros of the function $K(\mu)$. The series

$$F(z) \sim \sum_{\mu_\nu} Q_\nu(z) e^{\mu_\nu z}, \tag{24}$$

where

$$Q_\nu(z) e^{\mu_\nu z} = \frac{1}{2\pi i} \int_{C_\nu} \frac{\omega_K(\mu, \alpha, F)}{K(\mu)} e^{\mu z}\, d\mu.$$

corresponds to the function $F(z)$. Here C_ν is a circle with center at the point μ_ν within which there are no zeros of the function $K(\mu)$ that are different from μ_ν, and the function $\omega_K(\mu, \alpha, F)$ is determined by the formula

$$\omega_K(\mu, \alpha, F) = e^{-\alpha\mu} \int_{-qi}^{qi} \left[\int_0^{\xi} F(\alpha + \xi - \eta) e^{\mu\eta}\, d\eta \right] d\sigma(\xi). \tag{25}$$

If $F(z) \in C^n(a_1, b_1)$, the function (25) coincides with the previously introduced function $\omega_K(\mu, \alpha, F)$. This follows from the fact that, as is easy to verify, these functions are equal for functions of the form $F(z) = e^{\lambda z}$ with any λ. Let A and δ be fixed positive numbers. By D_1 we denote a rectangle $-A < x < 0$, $\operatorname{Im} a_1 + \delta < y < \operatorname{Im} b_1 - \delta$, and by D_2 a rectangle $0 < x < A$, $\operatorname{Im} a_1 + \delta < y < \operatorname{Im} b_1 - \delta$. We subject the previously introduced number β to the condition: $\sin \beta < \frac{(\sqrt{2}-1)\delta}{2A}$. We let S' denote the region lying to the left of the contour Γ_β formed by the rays $\arg \mu = \pm (\frac{\pi}{2} + \beta)$, and S'' the remaining region of the plane. It is shown in [1] that there exist limits

$$F_1(z) = \lim_{k \to \infty} \sum_{|\mu_\nu| < \rho_k,\ \mu_\nu \in S'} Q_\nu(z) e^{\mu_\nu z}, \quad z \in D_2, \tag{26}$$

$$F_2(z) = \lim_{k \to \infty} \sum_{|\mu_\nu| < \rho_k,\ \mu_\nu \in S''} Q_\nu(z) e^{\mu_\nu z}, \quad z \in D_1, \tag{27}$$

the convergence within these regions being uniform. It is proved that uniformly

for all y (Im a_1 + y Im b_1 - δ)

$$\lim_{x\to 0}[F_1(z)+F_2(\tilde{z})] = F(iy)\ (z = x+iy,\ \tilde{z} = -x+iy). \qquad (28)$$

This relation can be considered as the method of Abel for the summation of the generally diverging series (24). Let the function F(z) have a continuous derivative $F'(z)$ on (a_1,b_1). Let us verify that a series whose terms are derivatives of the terms of series (24) corresponds to the function $F'(z)$. By virtue of (25) we have

$$\omega_K(\mu,\alpha,F') = e^{-\alpha\mu}\int_{-ql}^{ql}\left[\int_0^{\xi}F'(\alpha+\xi-\eta)e^{\mu\eta}\,d\eta\right]d\sigma(\xi).$$

Since

$$\int_0^{\xi}F'(\alpha+\xi-\eta)e^{\mu\eta}\,d\eta = -F(\alpha)e^{\xi\mu}+F(\alpha+\xi)+\mu\int_0^{\xi}F(\alpha+\xi-\eta)e^{\mu\eta}\,d\eta,$$

we find that

$$\int_{-qi}^{qi}\left[\int_0^{\xi}F'(\alpha+\xi-\eta)e^{\mu\eta}\,d\eta\right]d\sigma(\xi) = -F(\alpha)K(\mu)+$$
$$+\int_{-qi}^{qi}F(\alpha+\xi)\,d\sigma(\xi)+\mu\int_{-ql}^{ql}\left[\int_0^{\xi}F(\alpha+\xi-\eta)e^{\mu\eta}\,d\eta\right]d\sigma(\xi).$$

The middle term on the right side of this equality equals zero. Therefore,

$$\frac{1}{2\pi i}\int_{C_\nu}\frac{\omega_K(\mu,\alpha,F')}{K(\mu)}e^{\mu z}\,d\mu = \frac{1}{2\pi i}\int_{C_\nu}\frac{\mu\omega_K(\mu,\alpha,F)}{K(\mu)}e^{\mu z}\,d\mu =$$
$$= \frac{d}{dz}\frac{1}{2\pi i}\int_{C_\nu}\frac{\omega_K(\mu,\alpha,F)}{K(\mu)}e^{\mu z}\,d\mu = \frac{d}{dz}[Q_\nu(z)e^{\mu z}],$$

which had to be proved. Hence it follows that relations of the form (26),(27) and (28) will also be valid for the derivative $F'(z)$. Taking this and theorem 4 into account, we formulate the following theorem.

<u>Theorem 5</u>. <u>Let the function</u> $\psi(z)$ <u>from the class</u> $C^n(a_1,b_1)$ <u>satisfy Eq.(1) for</u> $z \in (a_1 - a, b_1 - b)$ <u>and let series (5) correspond to the function</u> $\psi(z)$. <u>Then,</u>

$$\psi^{(m)}(iy) = \lim_{x\to 0}[\psi_1^{(m)}(z)+\psi_2^{(m)}(\tilde{z})],\quad iy\in(a_1,b_1)\ (m=0,1,\ldots,n), \qquad (29)$$

where

$$\psi_1^{(m)}(z) = \lim_{k\to\infty} \sum_{\substack{|\lambda_\nu| < \rho_k \\ \lambda_\nu \in S'}} [P_\nu(z) e^{\lambda_\nu z}]^{(m)}, \quad \psi_2^{(m)}(\tilde{z}) = \lim_{k\to\infty} \sum_{\substack{|\lambda_\nu| < \rho_k \\ \lambda_\nu \in S''}} [P_\nu(\tilde{z}) e^{\lambda_\nu \tilde{z}}]^{(m)}$$

and $z = x + iy$, $\tilde{z} = -x + iy$, $x > 0$. In relation (29) the convergence is uniform for $iy \in [\alpha, \beta] \subset (a_1, b_1)$.

As corollaries of theorem 5 we note the following propositions:

1) if $\psi(z) = 0$ on the segment $[a,b]$, then $\psi(z) = 0$ everywhere on (a_1, b_1);

2) the solution $\psi(z)$ in any segment $[\alpha, \beta] \subset (a_1, b_1)$ can be approximated as well as desired by means of linear finite combinations of functions from the system

$$z^s e^{\lambda_\nu z} \quad (s = 0, 1, \ldots, m_\nu - 1; \quad \nu = 1, 2, \ldots);$$

to be more precise, for any $\varepsilon > 0$ there is found an aggregate

$$P(z) = \sum_{\nu=1}^{N} \sum_{s=0}^{m_\nu - 1} a_{\nu,s} z^s e^{\lambda_\nu z},$$

satisfying the condition

$$|\psi^{(m)}(z) - P^{(m)}(z)| < \varepsilon, \quad z \in [\alpha, \beta] \quad (m = 0, 1, \ldots, n).$$

In [1] the following formula was established for the solution (24) of Eq.(21):

$$F(z) - \sum_{|\mu_\nu| < \rho} Q_\nu(z) e^{\mu_\nu z} =$$

$$= \frac{1}{2\pi i} \int_{|\mu| = \rho} \frac{1}{\mu^s K(\mu)} \left\{ \int_a^b \left[\int_0^\xi F^{(s)}(\xi - \eta + z) e^{\mu\eta} d\eta \right] d\sigma(\xi) \right\} d\mu,$$

where s is any number for which there exists a bounded piecewise continuous derivative $F^{(s)}(z)$. Hence

$$F^{(m)}(z) - \sum_{|\mu_\nu| < \rho} [Q_\nu(z) e^{\mu_\nu z}]^{(m)} =$$

$$= \frac{1}{2\pi i} \int_{|\mu| = \rho} \frac{\mu^m}{\mu^s K(\mu)} \left\{ \int_a^b \left[\int_0^\xi F^{(s)}(\xi - \eta + z) e^{\mu\eta} d\eta \right] d\sigma(\xi) \right\} d\mu. \tag{30}$$

Let the condition

$$|K(\mu)| > A|\mu|^{-p} e^{q|\sin\theta||\mu|}, \quad \theta = \arg\mu. \tag{31}$$

be fulfilled on a certain system of circles $|\mu| = \rho_k$ ($\rho_k \uparrow \infty$). Under this condition, as shown in [1], for z from the segment $[\alpha, \beta] \subset (a_1 + qi, b_1 - qi)$ and $|\mu| = \rho_k$ there holds the estimate

$$\left|\frac{1}{K(\mu)} \int_0^{\xi} F^{(s)}(\xi - \eta + z) e^{\mu\eta} d\eta\right| < \varepsilon\rho_k^p, \quad k > N(\varepsilon),$$

where $\varepsilon > 0$ is any number. Because of this fact, we find according to relations (30) and (31)

$$\left|F^{(m)}(z) - \sum_{|\mu_\nu| < \rho_k} [Q_\nu(z) e^{\mu_\nu z}]^{(m)}\right| < \frac{\varepsilon}{\rho_k^{s-m-p-1}}, \quad z \in [\alpha, \beta]. \tag{32}$$

We note that the function $K(\mu)$ has the form (15), where the numbers $n_1, \ldots, n_s$ can be chosen such that $n_1 + \ldots + n_s = n + 2$. Condition (31) will be fulfilled for $p = n - r + 2$ if

$$|L(\mu)| > B|\mu|^r e^{q|\sin\theta||\mu|}, \quad \theta = \arg\mu, \quad |\mu| = \rho_k, \quad \rho_k \uparrow \infty. \tag{33}$$

Taking inequality (32) into account and considering theorem 4, we get the following statement.

<u>Theorem 6</u>. <u>Let the function $\psi(z)$ have no fewer than</u> s ($s \geq n$) <u>derivatives on the interval</u> (a_1, b_1); <u>in this case, if</u> $s > n$, <u>then</u> $s - 1$ <u>first derivatives are continuous, and the derivative</u> $\psi^{(s)}(z)$ <u>is bounded piecewise-continuous. Let</u> $\psi(z)$ <u>satisfy Eq.(1) in the interval</u> $(a_1 - a, b_1 - b)$ <u>and let series</u> (5) <u>correspond to the function</u> $\psi(z)$. <u>If condition</u> (33) <u>is fulfilled, then</u>

$$\left|\psi^{(m)}(z) - \sum_{|\lambda_\nu| < \rho_k} [P_\nu(z) e^{\lambda_\nu z}]^{(m)}\right| < \frac{\varepsilon}{\rho_k^{s+r-n-m-3}}, \quad z \in [\alpha, \beta] \subset (a_1 + qi, b_1 - qi).$$

In conclusion let us give one more theorem, which was established in [1] for the solutions of Eq.(21); by virtue of theorem 4, it will also be valid for the solutions of Eq.(1).

Theorem 7. Let the function $\varphi(z)$ satisfy Eq. (1) on the whole axis and let series (5) correspond to the function $\varphi(z)$. If the function $\varphi(z)$ is regular on the segment $[a,b]$, then a certain subsequence of partial sums of series (5)

$$\sum_{|\lambda_\nu|<\rho_k} P_\nu(z)e^{\lambda_\nu z} \quad (k=1, 2, \ldots)$$

converges uniformly within a certain strip

$$-\infty \leqslant x_1 < \mathrm{Re}(z) < x_2 \leqslant \infty, \quad x_1 < 0 < x_2, \qquad (34)$$

moreover, it converges to the function $\varphi(z)$; consequently, the function $\varphi(z)$ is analytic in the strip (34).

Moscow

Received
March 16, 1964

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